High order finite element methods for wave propagation in heterogenous media

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Main motivations

We consider the following 1D toy problem

$$\left\{ \begin{array}{rcll} -\omega^2 u(x) - u''(x) & = & 1, & x \in (0,1), \\ u(0) & = & 0, & \\ u'(1) - i\omega u(1) & = & 0, & \end{array} \right.$$

whose solution is given by

$$u(x) = \frac{1}{\omega^2}(1 - \cos(\omega x)).$$

Main motivations

We discretize the problem with a finite element method of degree p.

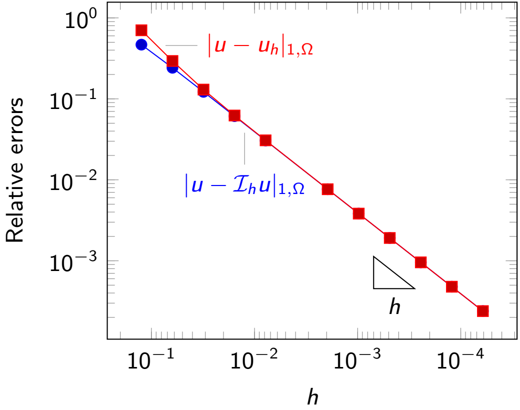
We plot convergence curves of the finite element solution $u_{h,p}$ and finite-element interpolant $\mathcal{I}_{h,p}u$ to the analytical solution u for different frequencies ω.

Main motivations

We can think of $\mathcal{I}_{h,p}u$ as an "ideal" representation of the solution u into the discretization space. However, it is not computable in general.

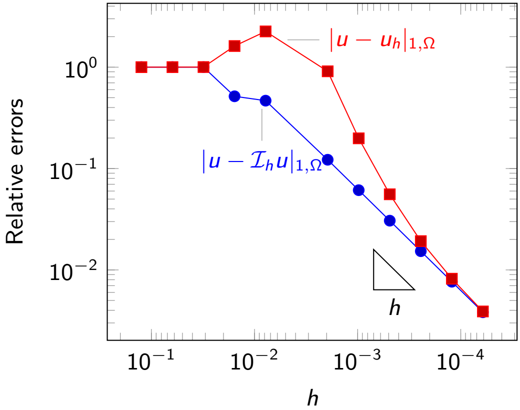
On the other hand, $u_{h,p}$ is obtained by solving a linear system. In some cases, we can "hope" that it is an approximation as good as $\mathcal{I}_{h,p}u$.

A low frequency case with $\mathcal{P}_1$ elements: $\omega = 4\pi$



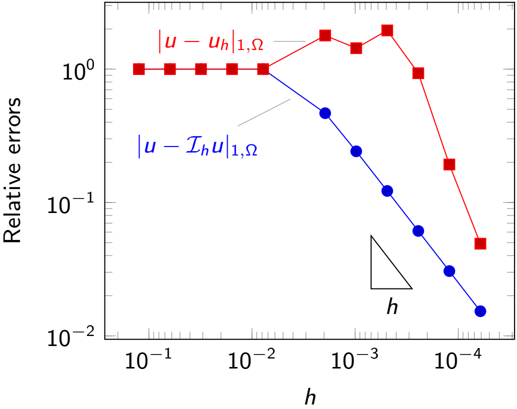
The behaviour FEM error and interpolation error are comparable.

A high frequency case with $\mathcal{P}_1$ elements: $\omega = 64\pi$



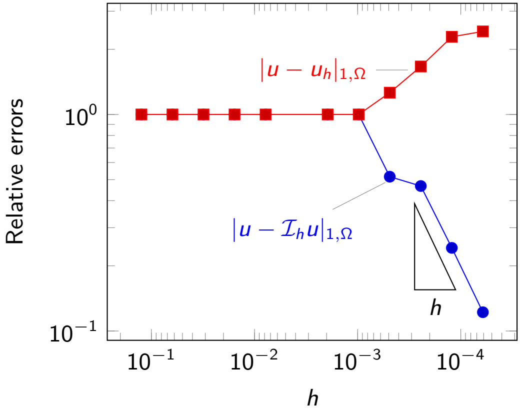
We observe a “gap” between the FEM and the interpolation error.

A high frequency case with $\mathcal{P}_1$ elements: $\omega = 256\pi$



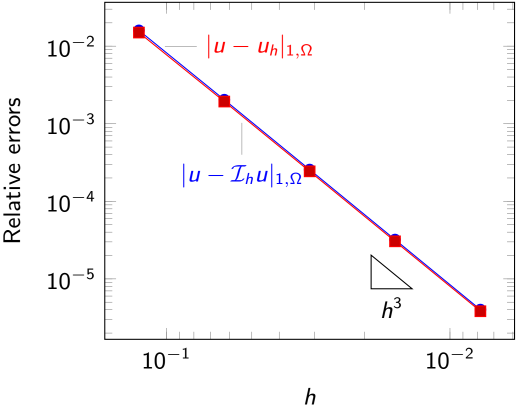
The "gap" is more important for this higher frequency.

A high frequency case with $\mathcal{P}_1$ elements: $\omega = 2048\pi$



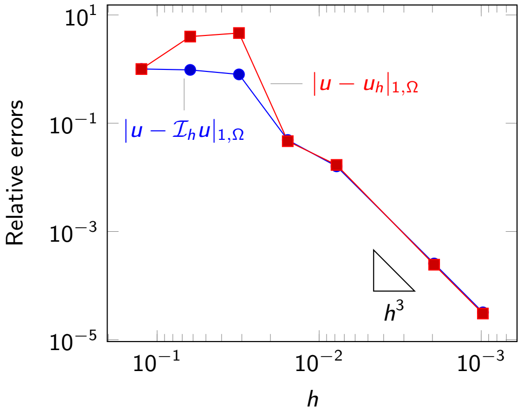
It looks like the FEM is not even converging!

A low frequency case with $\mathcal{P}_3$ elements: $\omega = 4\pi$



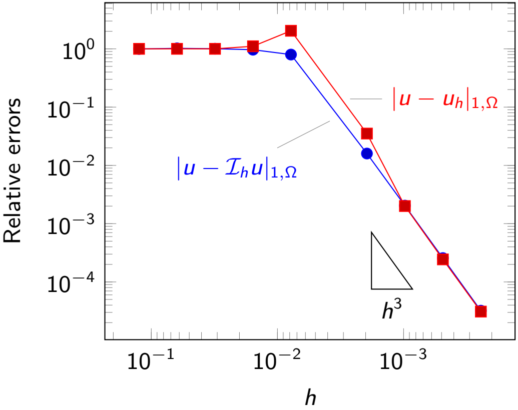
The behaviour FEM error and interpolation error are comparable.

A high frequency case with $\mathcal{P}_3$ elements: $\omega = 64\pi$



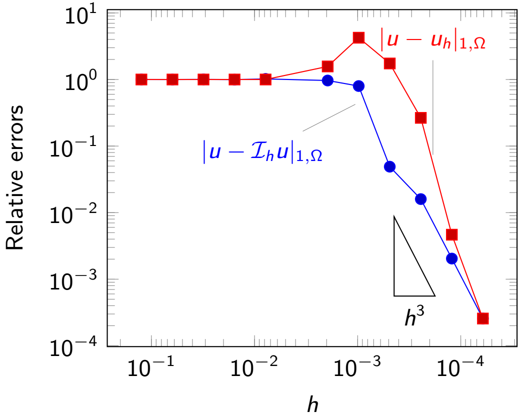
The FEM look stable.

A high frequency case with $\mathcal{P}_3$ elements: $\omega = 256\pi$



The FEM still look stable.

A high frequency case with $\mathcal{P}_3$ elements: $\omega = 2048\pi$



We finally see the gap.

Some observations

There is a “gap” between the interpolation and finite element errors.

This gap increases with the frequency.

This gap is less important for $p = 3$ than $p = 1$.

This is the manifestation of a lack of stability.
It is also called “pollution effect” in the literature.

Purpose of this talk

For the acoustic Helmholtz equation in homogeneous media, the pollution effect is well-understood:

F. Ihleburg I. Babuška, SIAM J. Numer. Anal., 2007.
J.M. Melenk, S.A. Sauter, Math. Comp., 2010.
J.M. Melenk, S.A. Sauter, SIAM J. Numer. Anal., 2011.

The purpose of this talk is to present a new approach to analyze the pollution effect in heterogeneous media and with more complex wave operators.

T. Chaumont-Frelet, S. Nicaise, IMA J. Numer. Anal., 2019.

Outline

1. Settings and key assumptions
2. Analysis of the pollution effect
3. Conclusions

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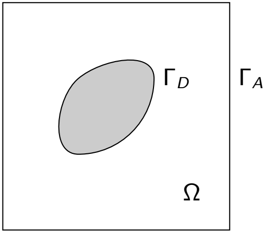
Settings

We consider the following time-harmonic wave propagation problem

$$\left\{ \begin{array}{rcll} -\omega^2 \mathcal{L}_0 u - \mathcal{L}_2 u & = & f & \text{in } \Omega \\ u & = & 0 & \text{on } \Gamma_D, \\ \mathcal{B} u - i\omega u & = & 0 & \text{on } \Gamma_A, \end{array} \right.$$



where:
Ω is a domain in $\mathbb{R}^d$, $d \in \{2, 3\}$
$\mathcal{L}_j$ is a differential operator of order j,
$\mathcal{B}$ is a neumann-type boundary operator associated with $\mathcal{L}_2$,
ω is the frequency.

Examples: Acoustic waves

We have

$$\begin{cases} -\dfrac{\omega^2}{\kappa}u - \nabla \cdot \left(\dfrac{1}{\rho}\nabla u\right) & = & f & \text{in } \Omega \\ u & = & 0 & \text{on } \Gamma_D, \\ \sqrt{\dfrac{\kappa}{\rho}}\nabla u \cdot \boldsymbol{n} - i\omega u & = & 0 & \text{on } \Gamma_A, \end{cases}$$

with

$$\mathcal{L}_0 v = \kappa^{-1} v, \quad \mathcal{L}_2 v = \nabla \cdot (\rho^{-1}\nabla v), \quad \mathcal{B}v = \sqrt{\kappa}\rho\nabla v \cdot \boldsymbol{n}.$$

Examples: Elastic waves

We have

$$\left\{\begin{array}{rcll} -\rho\boldsymbol{\omega}^2 u - \boldsymbol{\nabla}\cdot\sigma(u) & = & f & \text{in } \Omega, \\ u & = & 0 & \text{on } \Gamma_D \\ A\sigma(u)\boldsymbol{n} - i\boldsymbol{\omega} u & = & 0 & \text{on } \Gamma_A, \end{array}\right.$$

with

$$\mathcal{L}_0 v = \rho v, \quad \mathcal{L}_2 v = \boldsymbol{\nabla}\cdot\sigma(v), \quad \mathcal{B} v = A\sigma(u)\boldsymbol{n},$$

where A is smooth matrix depending on $\boldsymbol{n}$, ρ and σ.

Key assumptions: regularity

For $0 \leq m \leq p-1$, $f \in H^m(\Omega)$ and $g \in H^{m+1}(\Omega)$,
there exists a unique $v \in H^{m+2}(\Omega)$ such that

$$\left\{ \begin{array}{rcll} -\mathcal{L}_2 v & = & f & \text{in } \Omega, \\ v & = & 0 & \text{on } \Gamma_D, \\ \mathcal{B} v & = & g & \text{on } \Gamma_A, \end{array} \right.$$

and we have

$$|v|_{m+2,\Omega} \lesssim \|f\|_{m,\Omega} + \|g\|_{m+1,\Omega}.$$

We have

$$\|\mathcal{L}_0 v\|_{m,\Omega} \lesssim \|v\|_{m,\Omega}, \quad \forall v \in H^m(\Omega).$$

These hypothesis are typically satisfied as long as the boundaries and the heterogeneous coefficients are sufficiently smooth.

Up to some technicalities, we can weaken these assumption to treat piecewise smooth boundary and coefficients. We avoid it here for the sake of simplicity.

Key assumptions: stability

For all $f \in L^2(\Omega)$, there exists a unique $v \in H^1(\Omega)$ such that

$$\left\{ \begin{array}{rcll} -\omega^2 \mathcal{L}_0 v - \mathcal{L}_2 v & = & f & \text{in } \Omega, \\ v & = & 0 & \text{on } \Gamma_D, \\ \mathcal{B} v - i\omega v & = & 0 & \text{on } \Gamma_A, \end{array} \right.$$

and we have

$$|v|^2_{1,\Omega} \lesssim \|f\|_{0,\Omega}.$$

This hypothesis is satisfied in non-trapping domains, and corresponds to specific assumptions on the boundaries and coefficients.

We can actually weaken this hypothesis, but we avoid it here for the sake of simplicity.

T. Chaumont-Frelet, S. Nicaise, IMA J. Numer. Anal., 2019.

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Outline

Variational formulation

We denote by $b(\cdot,\cdot)$ the sesquilinear form associated with the boundary value problem.

Then, u is characterized as as the unique element $u \in H^1_{\Gamma_D}(\Omega)$ such that

$$b(u,v) = (f,v), \quad \forall v \in H^1_{\Gamma_D}(\Omega).$$

As usual, the FEM solution is defined by $u_{h,p} \in V_{h,p}$ such that

$$b(u_{h,p}, v_{h,p}) = (f, v_{h,p}), \quad \forall v \in V_{h,p},$$

where

$$V_{h,p} = \left\{ v_{h,p} \in H^1_{\Gamma_D}(\Omega) \mid v|_K \in \mathcal{P}_p(K) \; \forall K \in \mathcal{T}_h \right\}.$$

Gårdling inequality

We equip $H^1_{\Gamma_D}(\Omega)$ with the norm

$$\|v\|^2_{1,\omega,\Omega} = \omega^2 \|v\|^2_{0,\Omega} + |v|^2_{1,\Omega}.$$

The sesquilinear form $b(\cdot,\cdot)$ is continuous

$$|b(u,v)| \lesssim \|u\|_{1,\omega,\Omega} \|v\|_{1,\omega,\Omega},$$

but it is not coercive.

It satisfies a (weakest) Gårding inequality:

$$\operatorname{Re} b(v,v) \gtrsim \|v\|^2_{1,\omega,\Omega} - \omega^2 \|v\|^2_{0,\Omega}..$$

The “non-coerciveness” is more important for high frequencies.

Outline

Duality analysis: the Schatz argument

We cannot apply Céa's Lemma directly.

We want to get rid of the negative $L^2(\Omega)$-term in Gårding inequality.

We can do it by duality, using the so-called "Schatz argument"
A.H. Schatz, Math. Comp., 1974.
F. Ihlenburg, I. Babuška, Comp. Math. Appl., 1995.

This method employs the well-known Aubin-Nitsche trick.

The Aubin-Nitsche trick

We introduce $\xi \in H^1_{\Gamma_D}(\Omega)$ such that

$$b(w,\xi) = (w, u - u_{h,p}) \quad \forall w \in H^1_{\Gamma_D}(\Omega).$$

We pick the test function $w = u - u_{h,p}$ and employ Galerkin's orthogonality:

$$\|u-u_{h,p}\|^2_{0,\Omega} = b(u-u_{h,p},\xi) = b(u-u_{h,p},\xi-\mathcal{I}_{h,p}\xi) \lesssim \|u-u_{h,p}\|_{1,\boldsymbol{\omega},\Omega}\|\xi-\mathcal{I}_{h,p}\xi\|_{1,\boldsymbol{\omega},\Omega}.$$

On the other hand, we have

$$\|\xi - \mathcal{I}_{h,p}\xi\|_{1,\boldsymbol{\omega},\Omega} \lesssim h|\xi|_{2,\Omega} \lesssim \boldsymbol{\omega} h\|u - u_{h,p}\|_{0,\Omega},$$

where we have employed the stability assumption.

Aubin-Nitsche trick

It follows that

$$\|u - u_{h,p}\|_{0,\Omega} \lesssim \omega h \|u - u_{h,p}\|_{1,\omega,\Omega},$$

and thus

$$\begin{aligned} b(u - u_{h,p}, u - u_{h,p}) &\gtrsim \|u - u_{h,p}\|^2_{1,\omega,\Omega} - \omega^2 \|u - u_{h,p}\|^2_{0,\Omega} \\ &\gtrsim \left(1 - \omega^4 h^2\right) \|u - u_{h,p}\|^2_{1,\omega,\Omega} \\ &\gtrsim \|u - u_{h,p}\|^2_{1,\omega,\Omega} \end{aligned}$$

under the assumption that $\omega^2 h$ is small enough.

A first conclusion

Using Galerkin's orthogonality, it follows that if $\omega^2 h$ is small enough, then

$$\begin{aligned}
\|u - u_{h,p}\|^2_{1,\omega,\Omega} &\lesssim b(u - u_{h,p}, u - u_{h,p}) \\
&= b(u - u_{h,p}, u - \mathcal{I}_{h,p}u) \\
&\lesssim \|u - u_{h,p}\|_{1,\omega,\Omega} \|u - \mathcal{I}_{h,p}u\|_{1,\omega,\Omega},
\end{aligned}$$

and

$$\|u - u_{h,p}\|_{1,\omega,\Omega} \lesssim \|u - \mathcal{I}_{h,p}u\|_{1,\omega,\Omega}.$$

Thus, the FEM is quasi-optimal asymptotically, under the condition that $\omega^2 h \lesssim 1$.

A first conclusion

The condition that $\omega^2 h \lesssim 1$ is very restrictive if the frequency is high.

Numerical experiments indicate that high order FEM seem more stable, but this quasi-optimality condition does not reflect it.

We need to provide a sharper analysis.

Outline

Getting back to the Schatz argument

In the Schatz argument, we employed the function $\xi \in H^1_{\Gamma_D}(\Omega)$ solution to

$$b(w,\xi) = (w, u - u_{h,p}) \quad \forall w \in H^1_{\Gamma_D}(\Omega).$$

Then, the key steps of the proof were to show that

$$\|u - u_{h,p}\|_{0,\Omega} \lesssim \|u - u_{h,p}\|_{1,\omega,\Omega} \|\xi - \mathcal{I}_{h,p}\xi\|_{1,\omega,\Omega},$$

and

$$\|\xi - \mathcal{I}_{h,p}\xi\|_{1,\omega,\Omega} \lesssim \omega h \|u - u_{h,p}\|_{0,\Omega}.$$

Actually, the second estimate is not sharp when p is high.

The interpolation error

To analyze the case of high order FEM, let us introduce

$$\eta = \sup_{\phi \in L^2(\Omega) \setminus \{0\}} \frac{\|u_\phi^\star - \mathcal{I}_{h,p} u_\phi^\star\|_{1,\omega,\Omega}}{\|\phi\|_{0,\Omega}},$$

where $u_\phi^\star$ is the unique element of $H^1_{\Gamma_D}(\Omega)$ such that

$$b(w, u_\phi^\star) = (w, \phi) \quad \forall \phi \in H^1_{\Gamma_D}(\Omega).$$

By definition of η, we have

$$\|u_\phi^\star - \mathcal{I}_{h,p} u_\phi^\star\|_{1,\omega,\Omega} \leq \eta \|\phi\|_{0,\Omega}.$$

Quasi-optimality of the FEM

We remark that we have in particular

$$\|\xi - \mathcal{I}_{h,p}\xi\|_{1,\omega,\Omega} \leq \eta \|u - u_{h,p}\|_{0,\Omega},$$

so that

$$\|u - u_{h,p}\|_{0,\Omega} \lesssim \eta \|u - u_{h,p}\|_{1,\omega,\Omega}.$$

We thus have

$$\operatorname{Re} b(u - u_{h,p}, u - u_{h,p}) \gtrsim \left(1 - \omega^2\eta^2\right) \|u - u_{h,p}\|_{1,\Omega}^2,$$

and we conclude that the FEM is quasi-optimal if $\omega\eta \lesssim 1$.

As a result we need to derive a sharp estimate for the quantity η.

Outline

A regularity issue

Take $\phi \in L^2(\Omega)$ and define $u_\phi^\star$ as the solution to

$$\left\{ \begin{array}{rcll} -\omega^2 \mathcal{L}_0 u_\phi^\star - \mathcal{L}_2 u_\phi^\star & = & \phi & \text{in } \Omega, \\ u_\phi^\star & = & 0 & \text{on } \Gamma_D, \\ \mathcal{B} u_\phi^\star - i\omega u_\phi^\star & = & 0 & \text{on } \Gamma_A. \end{array} \right.$$

Because $\phi \in L^2(\Omega)$ only, we only have $u_\phi^\star \in H^2(\Omega)$.

Thus, it appears complicated to take advantage of high order polynomials, since they require additional regularity to provide improved convergence rates.

Regularity splitting: the main idea

Since we cannot hope more that $H^2(\Omega)$ regularity, the asymptotic convergence rate is in $\mathcal{O}(h)$.

In addition, we cannot assume more regularity on the datum ϕ, since the estimate on $\boldsymbol{\eta}$ is employed in a duality argument.

So the main idea is to split the solution into two parts:
one that is only not regular but behaves well when the frequency is high,
another that is highly oscillatory, but smooth.

J.M. Melenk, S.A. Sauter, Math. Comp., 2010.
J.M. Melenk, S.A. Sauter, SIAM J. Numer. Anal., 2011.

A kind of asymptotic expansion

Our main idea to obtain the regularity splitting is to formally expend $u_\phi^\star$ as

$$u_\phi^\star = u_0 + \omega u_1 + \omega^2 u_2 + \omega^3 u_3 + \dots$$

and “hope” that the iterates have increasing regularity.

Recurrence relations

Formally inserting the expansion into the wave propagation problem, we obtain the following definitions:

$$\left\{\begin{array}{rcll} -\mathcal{L}_2 u_0 & = & \phi & \text{in } \Omega, \\ u_0 & = & 0 & \text{on } \Gamma_D, \\ \mathcal{B} u_0 & = & 0 & \text{on } \Gamma_A, \end{array}\right.$$

$$\left\{\begin{array}{rcll} -\mathcal{L}_2 u_1 & = & 0 & \text{in } \Omega, \\ u_1 & = & 0 & \text{on } \Gamma_D, \\ \mathcal{B} u_1 & = & u_0 & \text{on } \Gamma_A, \end{array}\right.$$

and

$$\left\{\begin{array}{rcll} -\mathcal{L}_2 u_{n+2} & = & \mathcal{L}_0 u_n & \text{in } \Omega, \\ u_{n+2} & = & 0 & \text{on } \Gamma_D, \\ \mathcal{B} u_{n+2} & = & u_{n+1} & \text{on } \Gamma_A. \end{array}\right.$$

Regularity of the iterates

Using the regularity assumption, we easily obtain that

$$u_n \in H^{n+2}(\Omega), \quad \|u_n\|_{n+2,\Omega} \lesssim \|\phi\|_{0,\Omega}$$

by employing an induction argument.

Then, we write

$$u_\phi^\star = \sum_{j=0}^{p-2} \omega^j u_j + r_p,$$

and we obtain another recurrence relation for the sequence (r_p).

Regularity of the remainder

Using the recurrence relation on (r_p), we show that

$$r_p \in H^{p+1}(\Omega), \quad \|r_p\|_{p+1,\Omega} \lesssim \omega^{p+1} \|\phi\|_{0,\Omega}.$$

Here, we also employed the stability assumption.

Estimation of η

We have

$$u_\phi^\star = \sum_{j=0}^{p-2} \omega^j u_j + r_p,$$

and therefore

$$\begin{aligned}
\|u_\phi^\star - \mathcal{I}_{h,p} u_\phi^\star\|_{1,\omega,\Omega} &\lesssim \sum_{j=0}^{p-2} \omega^j \|u_j - \mathcal{I}_{h,p} u_j\|_{1,\omega,\Omega} + \|r_p - \mathcal{I}_{h,p} r_p\|_{1,\omega,\Omega} \\
&\lesssim \sum_{j=0}^{p-2} \omega^j h^{j+1} \|u_j\|_{j+2,\Omega} + h^p \|r_p\|_{p+1,\Omega} \\
&\lesssim \sum_{j=0}^{p-2} \omega^j h^{j+1} \|\phi\|_{0,\Omega} + \omega^p h^p \|\phi\|_{0,\Omega} \\
&\lesssim \left(h \left(\sum_{j=0}^{p-2} \omega^j h^j \right) + \omega^p h^p \right) \|\phi\|_{0,\Omega} \\
&\lesssim (h + \omega^p h^p) \|\phi\|_{0,\Omega}
\end{aligned}$$

Estimation of η

We have thus established that

$$\|u_\phi^\star - \mathcal{I}_{h,p} u_\phi^\star\|_{1,\omega,\Omega} \lesssim (h + \omega^p h^p) \|\phi\|_{0,\Omega}$$

for all $\phi \in L^2(\Omega)$.

Recalling that

$$\eta = \sup_{\phi \in L^2(\Omega)\setminus\{0\}} \frac{\|u_\phi^\star - \mathcal{I}_{h,p} u_\phi^\star\|_{1,\omega,\Omega}}{\|\phi\|_{0,\Omega}},$$

we see that

$$\eta \lesssim h + \omega^p h^p.$$

Outline

Main result

We recall that the FEM is quasi-optimal under the condition that $\omega\eta \lesssim 1$.

The condition $\omega\eta \lesssim 1$ is thus equivalent to

$$\omega^{p+1}h^p \lesssim 1,$$

that we may rewrite

$$h \lesssim \omega^{-1-1/p}.$$

Recalling that the number of elements per wavelength satisfies

$$N_\lambda \simeq (\omega h)^{-1},$$

we see that this condition is equivalent to

$$N_\lambda \gtrsim \omega^{1/p}.$$

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Conclusions

The FEM is quasi-optimal under the condition that

$$\omega^{p+1} h^p \lesssim 1.$$

Numerical experiments show that these conditions are sharp.

To preserve the stability of the FEM, the number of elements per wavelength must be increased when the frequency is high

$$N_\lambda \gtrsim \omega^{1/p}.$$

However, this effect is much less important for high order methods, as the increased rate is $1/p$.

Conclusions

These results encourage the use of high order FEM (or DGM) to solve high-frequency problems.

The final results are very similar to previous works obtained for acoustic wave propagation in homogeneous media.

The main novelty is that heterogeneous media, and more general wave propagation operators are treated.